

ON POLYA'S THEOREM IN SEVERAL COMPLEX VARIABLES

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ABSTRACT. Let K be a compact set in \mathbb{C} , f a function analytic in $\overline{\mathbb{C}} \setminus K$ vanishing at ∞ . Let $f(z) = \sum_{k=0}^{\infty} a_k z^{-k-1}$ be its Taylor expansion at ∞ , and $H_s(f) = \det(a_{k+l})_{k,l=0}^s$ the sequence of Hankel determinants. The classical Polya inequality says that

$$\limsup_{s \rightarrow \infty} |H_s(f)|^{1/s^2} \leq d(K),$$

where $d(K)$ is the transfinite diameter of K . Goluzin has shown that for some class of compacta this inequality is sharp. We provide here a sharpness result for the multivariate analog of Polya's inequality, considered by the second author in Math. USSR Sbornik, 25 (1975), 350-364.

1. PRELIMINARIES AND INTRODUCTION

We denote by $A(\mathbb{C}^n)^*$ the dual space to the space $A(\mathbb{C}^n)$ of all entire functions on \mathbb{C}^n , equipped with the locally convex topology of locally uniform convergence in \mathbb{C}^n . Following Hörmander ([11], Section 4.5), we call the elements of $A(\mathbb{C}^n)^*$ *analytic functionals*.

Let \mathbb{Z}_+^n be the collection of all n -dimensional vectors with non-negative integer coordinates. For $k = (k_1, \dots, k_n) \in \mathbb{Z}_+^n$ and $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, let $z^k = z_1^{k_1} \dots z_n^{k_n}$ and $|k| := k_1 + \dots + k_n$ be the degree of the monomial z^k . We consider the enumeration $\{k(i)\}_{i \in \mathbb{N}}$ of the set \mathbb{Z}_+^n such that $|k(i)| \leq |k(i+1)|$ and on each set $\{|k(i)| = s\}$ the enumeration coincides with the lexicographic order relative to k_1, \dots, k_n . We will write $s(i) := |k(i)|$. The number of multiindices of degree at most s is $m_s := C_{s+n}^s$ and the number of those of degree exactly s is $N_s = m_s - m_{s-1} = C_{n+s-1}^s$, $s \geq 1$; $N_0 = 1$. Let $l_s := \sum_{q=0}^s q N_q$ for $s = 0, 1, 2, \dots$.

Consider Vandermondians:

$$V(\zeta_1, \dots, \zeta_i) := \det(e_\alpha(\zeta_\beta))_{\alpha, \beta=1}^i, \quad i \in \mathbb{N},$$

where $e_\alpha(z) := z^{k(\alpha)}$, $\alpha \in \mathbb{N}$ and $(\zeta_\beta) \in K^i$.

For a compact set $K \subset \mathbb{C}^n$, define "maximal Vandermondians":

$$V_i := \sup \{|V(\zeta_1, \dots, \zeta_i)| : (\zeta_j) \in K^i\}, \quad i \in \mathbb{N}.$$

Set $d_s(K) := (V_{m_s})^{1/l_s}$. The transfinite diameter of K is the number:

$$(1.1) \quad d(K) := \limsup_{s \rightarrow \infty} d_s(K).$$

In the one-dimensional case, this notion was introduced by Fekete [7] for $n = 1$, and by Leja [14] for $n \geq 2$. That, in fact, the usual limit can be taken in (1.1) was proved in [7] for $n = 1$ and in [25] for $n \geq 2$.

The *pluripotential Green function* of a compact set $K \subset \mathbb{C}^n$ is defined as follows

$$g_K(z) = \limsup_{\zeta \rightarrow z} \sup\{u(\zeta) : u|_K \leq 0, u \in \mathcal{L}(\mathbb{C}^n)\},$$

where $\mathcal{L}(\mathbb{C}^n)$ represents the Lelong class consisting of all functions $u \in Psh(\mathbb{C}^n)$ such that $u(\zeta) - \ln|\zeta|$ is bounded from above near infinity. Since $d(K) = d(\hat{K})$, with no loss of generality, we are going to consider polynomially convex compact sets. We will also consider the class of functions $\mathcal{L}^+(\mathbb{C}^n) := \{u \in \mathcal{L}(\mathbb{C}^n) : u(z) \geq \log^+|z| + C\}$. The function $g_K(z)$ is either plurisubharmonic in \mathbb{C}^n or identically equal to $+\infty$. For more detail about the pluripotential Green function, we refer the reader to [12], [20] and [26].

The *Monge-Ampere energy* $\mathcal{E}(u, v)$ of u relative to v for $u, v \in \mathcal{L}^+(\mathbb{C}^n)$ is defined as follows ([5], Section 5):

$$\mathcal{E}(u, v) := \int_{\mathbb{C}^n} (u - v) \sum_{j=0}^n (dd^c u)^j \wedge (dd^c v)^{n-j}.$$

Let K be a compact set in \mathbb{C}^n . $A(K)$ represents the locally convex space of all germs of analytic functions on K , equipped with the countable inductive limit topology, i.e.,

$$A(K) = \lim_{j \rightarrow \infty} \text{ind } A(D_j)$$

considered in regard to the inclusion of sets. D_j are open sets such that $D_{j+1} \Subset D_j$ for each $j \in \mathbb{N}$ and $K = \bigcap_{j=1}^{\infty} D_j$. Thus, in this setting, a sequence $\{u_j\}$ of germs converges to a germ u in this topology in case there exists an open neighbourhood $V \supset K$ and functions $g_j, g \in A(V)$ being the representatives of the germs u_j, u respectively, such that g_j converges uniformly to g on any compact subset of V .

The Polya Theorem (Theorem 2.1) and its multivariate analog (Theorem 2.3), considered by the second author in [25], are discussed in Section 2. The sharpness result of the generalized Polya inequality (section 4) is based on the comparison of the expression (2.6) for Hankel-like determinants from [25] with the expression (2.9) for the transfinite diameter from Bloom and Levenberg [1]. The main result of this article (Theorem 4.7) says that, for *real* compact sets, the equality *attains* in the generalized Polya inequality (2.7) for some analytic functional $f^* \in A(K)^*$. This result seems to be new even in the one-dimensional case. Additionally, we introduce two sharpness properties for compact sets $K \subseteq \mathbb{C}^n$ and study the stability of these properties relative to the approximations from inside and outside (Proposition 4.3 and 4.4).

2. POLYA'S THEOREM

The following result is due to Polya [19].

Theorem 2.1. *Let K be a polynomially convex compact set in \mathbb{C} and $f \in A(\overline{\mathbb{C}} \setminus K)$ have the following expansion in a neighbourhood of ∞ :*

$$(2.1) \quad f(z) = \sum_{k=0}^{\infty} \frac{a_k}{z^{k+1}}.$$

Let $A_s(f) := \det(a_{k+m})_{k,m=0}^{s-1}$, $s \in \mathbb{N}$, be a sequence of Hankel determinants composed from the coefficients of the expansion (2.1). Then,

$$(2.2) \quad D(f) := \limsup_{s \rightarrow \infty} |A_s(f)|^{1/s^2} \leq d(K).$$

A direct multivariate analog of the inequality (2.2) makes no sense, since there are functions analytic on the complement of K but constants only. Schiffer and Siciak ([21]) proved some analog for the product of plane compact sets $K = K_1 \times K_2 \times \dots \times K_n \subset \mathbb{C}^n$ and functions $f \in A((\overline{\mathbb{C}} - K_1) \times \dots \times (\overline{\mathbb{C}} - K_n))$. Sheinov ([22], [23]) considered another analog of Polya's inequality for a *linearly convex* compact set K , considering the Taylor expansion at the origin for functions analytic in the domain $D = K^*$ linearly convex adjoint (conjugate) to K (projective complement of K by Martineau [18]).

The case of an arbitrary compact set $K \subset \mathbb{C}^n$ was studied in [25]. It was suggested there, instead of analytic functions on some artificial "complement" of K , to deal with those analytic functionals in \mathbb{C}^n that are extendible continuously onto the space $A(\hat{K})$. We denote by $A_0(\{\infty^n\})$ the space of all analytic germs f' at $\infty^n = (\infty, \infty, \dots, \infty) \in \overline{\mathbb{C}^n}$ with Taylor expansion of the form

$$(2.3) \quad f'(z) = \sum_{k \in \mathbb{Z}^n} \frac{a_k}{z^{k+I}}, \quad I = (1, 1, \dots, 1),$$

converging in some neighborhood of ∞^n .

Lemma 2.2. *There is an isomorphism,*

$$(2.4) \quad T : A(\mathbb{C}^n)^* \rightarrow A_0(\{\infty^n\}),$$

such that, for each f^* and $f' = Tf^*$, we have

$$f^*(\varphi) = \langle \varphi, f' \rangle := \left(\frac{1}{2\pi i} \right)^n \int_{\mathbb{T}_R^n} \varphi(\zeta) f'(\zeta) d\zeta, \quad \varphi \in A(\mathbb{C}^n),$$

where

$$(2.5) \quad \mathbb{T}_R^n := \{z = (z_\nu) \in \mathbb{C}^n : |z_\nu| = R, \nu = 1, \dots, n\}, \quad R = R(f^*).$$

Proof. See, e.g., [6], Chapter 3. □

Let us define, for every analytic functional f^* , a related sequence of multivariate Hankel-like determinants constructed from the coefficients of the expansion (2.3):

$$(2.6) \quad H_i = H_i(f^*) := \det(a_{k(\alpha)+k(\beta)})_{\alpha, \beta=1}^i, \quad i \in \mathbb{N}$$

with $a_{k(\alpha)} := f^*(e_\alpha) = \langle e_\alpha, f' \rangle$, $\alpha \in \mathbb{N}$, $f' = Tf^*$. Now we are ready to formulate the general form of multivariate Polya's inequality.

Theorem 2.3. *Suppose that K is a compact set in \mathbb{C}^n , f^* is an analytic functional which has a continuous extension onto $A(K)$ and $f' = Tf^*$ is the corresponding analytic germ at ∞^n . Then for the determinants (2.6), the inequality holds:*

$$(2.7) \quad D(f') := \limsup_{i \rightarrow \infty} |H_i(f^*)|^{\frac{1}{2^{l_{s(i)}}}} \leq d(K).$$

It has been proved in [25] a bit weaker result with the outer transfinite diameter $\widehat{d}(K)$ instead of $d(K)$, but later it was proved that $\widehat{d}(K) = d(K)$ (see Proposition 3.1 below).

We send the reader for the proof of Theorem 2.3 to [25], Theorem 3. However we cite here the following equality, which is crucial there and will be essentially used in Section 4:

$$(2.8) \quad i! |H_i(f^*)| = |f_{\zeta^{(i)}}^*(\dots f_{\zeta^{(j)}}^*(\dots (f_{\zeta^{(1)}}^*([V(\zeta^{(1)}, \zeta^{(2)}, \dots, \zeta^{(i)})]^2) \dots) \dots))|,$$

$i \in \mathbb{N}$, here the notation $f_{\zeta^{(j)}}^*$ means that the functional f^* is applied sequentially to a function of the variable $\zeta^{(j)}$ by keeping the other variables fixed.

Remark 2.4. The classical Polya's Theorem (Theorem 2.1) is a particular case of Theorem 2.3 since, due to Gröthendieck-Köthe-Silva duality (see [10], [13], [24]), every $f \in A(\overline{\mathbb{C}} \setminus K)$ satisfying (2.1) in a neighborhood of ∞ represents a linear continuous functional $f^* \in A(K)^* \hookrightarrow A(\mathbb{C})^*$. Hereafter \hookrightarrow denotes a linear continuous embedding.

Let $K \subset \mathbb{C}^n$ be a compact set, and μ be a bounded positive Borel measure on K . The pair (K, μ) is said to satisfy *Bernstein-Markov inequality* for holomorphic polynomials in \mathbb{C}^n if, given $\varepsilon > 0$, there exists a constant $M = M(\varepsilon)$ such that for all polynomials p_s of degree at most s

$$\|p_s\|_K \leq M(1 + \varepsilon)^s \|p_s\|_{L^2(\mu)}.$$

Theorem 2.5. (*Bloom-Levenberg*, [1]) *Let $K \subset \mathbb{C}^n$ be a compact set, μ be a bounded positive Borel measure on K and let (K, μ) satisfy Bernstein-Markov inequality. Then,*

$$\lim_{s \rightarrow \infty} Z_s(K, \mu)^{\frac{1}{2I_s(n)}} = d(K),$$

where

$$(2.9) \quad Z_s(K, \mu) = \int_{K^{m_s(n)}} |V(\zeta^{(1)}, \dots, \zeta^{(m_s(n))})|^2 d\mu(\zeta^{(1)}) d\mu \dots d\mu(\zeta^{(m_s(n))}).$$

Remark 2.6. In [2] (Proposition 3.4 and Corollary 3.5), the same authors proved that for any compact set $K \subseteq \mathbb{C}^n$, there exists a measure $\mu \in \mathcal{M}(K)$ such that (K, μ) satisfies Bernstein-Markov property.

3. STABILITY OF TRANSFINITE DIAMETER

The following proposition provides the stability of transfinite diameter of a compact set in \mathbb{C}^n approximated from outside.

Proposition 3.1. (V.A. Znamenskii [29, 30], Levenberg [15]) *Let K be a compact set in \mathbb{C}^n and $\{K_j\}$ a sequence of compact sets such that $K_{j+1} \subseteq K_j$ for all $j \in \mathbb{N}$ and $K = \bigcap_{j=1}^{\infty} K_j$. Then,*

$$\widehat{d}(K) := \lim_{j \rightarrow \infty} d(K_j) = d(K).$$

In this section, we prove a stability property of transfinite diameter relative to the approximation from inside. The following is an easy consequence of Lemma 6.5 in [4]:

Lemma 3.2. *Suppose that K is a non-pluripolar compact set in \mathbb{C}^n , and $\{K_j\}$ is a sequence of non-pluripolar compact sets such that $K_j \subseteq K_{j+1} \subseteq K$, $j \in \mathbb{N}$ and for $L := \bigcup_{j=1}^{\infty} K_j$, we have*

$$(3.1) \quad \int_{K \setminus L} (dd^c g_K)^n = 0.$$

Then

$$\lim_{j \rightarrow \infty} g_{K_j}(z) = g_K(z), \quad z \in \mathbb{C}^n.$$

Theorem 3.3. *Under the conditions of Lemma 3.2, we have,*

$$\lim_{j \rightarrow \infty} d(K_j) = d(K).$$

Proof. We will use the unweighted energy version of Rumely's formula (see e.g., Theorem 5.1 of [16], or Section 9.1 of [5]). Since, by Lemma 3.2, $g_{K_j} \downarrow g_K$, applying the remark after Lemma 3.5 in [16], one obtains

$$-\ln d(K_j) = \frac{1}{n(2\pi)^n} \mathcal{E}(g_{K_j}, g_T) \uparrow \frac{1}{n(2\pi)^n} \mathcal{E}(g_K, g_T) = -\ln d(K), \text{ as } j \rightarrow \infty,$$

where T is the unit torus in \mathbb{C}^n . \square

4. SHARPNESS OF POLYA'S INEQUALITY

The following Theorem is proved by Goluzin in [8] (see also [9], Section 11).

Theorem 4.1. *For functions which are analytic in an infinite domain B with boundary K consisting of a finite number of closed Jordan curves and having the expansion*

$$(4.1) \quad f(z) = \sum_{k=1}^{\infty} \frac{a_k}{z^k},$$

in a neighborhood of $z = \infty$, the inequality $D(f) = \limsup_{s \rightarrow \infty} |A_s(f)|^{1/s^2} \leq d(K)$ given by Theorem 2.1 is sharp.

Another way of expressing Theorem 4.1 is, for a compact set $K \subseteq \mathbb{C}$

$$(4.2) \quad d(K) = \sup \{D(f) : f \in A(\overline{\mathbb{C}} \setminus K)\},$$

if the boundary ∂K consists of a finite number of closed Jordan curves.

Definition 4.2. Let K be a polynomially convex compact set in \mathbb{C}^n . K is said to satisfy the *sharpness property* in Polya inequality, shortly denoted as $K \in (SP)$, if

$$d(K) = \sup \{D(f') : f' = T(f^*), f^* \in A(K)^*\}.$$

We say that K has a *strong sharpness property* in Polya inequality, denoted by $K \in (SSP)$, if there exists a $f^* \in A(K)^*$ such that

$$D(f') = d(K)$$

for $f' = T(f^*)$, where T is defined as in Lemma 2.2.

If K is a pluripolar compact set in \mathbb{C}^n , then $K \in (SSP)$ by the result of Levenberg-Taylor ([17]) which says that $d(K) = 0$ if and only if K is pluripolar. From now on, we only consider non-pluripolar compact sets.

Proposition 4.3. Let K be a compact set in \mathbb{C}^n , $\{K_i\}$ a sequence of compact sets with $K = \bigcap_{i=1}^{\infty} K_i$. Assume $K_i \in (SP)$ for all $i \in \mathbb{N}$. Then there exists a sequence of analytic functionals $\{f_i^*\}$ such that $f_i^* \in A(K_i)^*$ for each $i \in \mathbb{N}$ and

$$(4.3) \quad \lim_{i \rightarrow \infty} D(f_i') = d(K).$$

Proof. By Definition 4.2, for each $i \in \mathbb{N}$, there exists $f_i^* \in A(K_i)^*$ with $f_i' = T(f_i^*)$ such that $d(K_i) \leq D(f_i') + \frac{1}{i}$. Theorem 2.3 gives $D(f_i') \leq d(K_i)$. By using Proposition 3.1, we have

$$d(K) = \lim_{i \rightarrow \infty} d(K_i) \leq \lim_{i \rightarrow \infty} D(f_i') \leq \lim_{i \rightarrow \infty} d(K_i) = d(K),$$

which gives the limit (4.3). \square

As seen from Proposition 4.3, (SP) is not preserved under the approximation from outside, however, for an approximation from inside, we have the stability of the property (SP) :

Proposition 4.4. Let the conditions of Lemma 3.2 be given. Suppose further that $K_i \in (SP)$ for all $i \in \mathbb{N}$. Then $K \in (SP)$.

Proof. Proof is almost the same as the proof of Proposition 4.3 except we only use Theorem 3.3 instead of Proposition 3.1 in the end, hence we have the following:

$$d(K) \leq \lim_{i \rightarrow \infty} D(f_i') = \sup\{D(f_i') : i \in \mathbb{N}\} \leq d(K),$$

which concludes that $d(K) = \sup\{D(f_i') : i \in \mathbb{N}\}$ and so $K \in (SP)$ by Definition 4.3. \square

For an arbitrary compact set in \mathbb{C} , a following sharpness statement, which is weaker than (SP) , is derived easily from Goluzin's result above.

Proposition 4.5. Let K be a compact set in \mathbb{C} , $\{K_i\}$ a sequence of compact sets with the properties $K_{i+1} \Subset K_i$ for all $i \in \mathbb{N}$, $K = \bigcap_{i=1}^{\infty} K_i$. Then there exists a sequence of functions $f_i \in A(\overline{\mathbb{C}} \setminus K_i)$ such that

$$(4.4) \quad \lim_{i \rightarrow \infty} D(f_i) = d(K).$$

Proof. For each $i \in \mathbb{N}$, we can find a compact set L_i whose boundary consists of a finite number of closed analytic Jordan curves so that $K_{i+1} \Subset L_i \Subset K_i$ holds. By the result of Goluzin, there exists $f_i \in A(\overline{\mathbb{C}} \setminus L_i)$ such that, $d(L_i) < D(f_i) + \frac{1}{i}$. Since $f_i \in A(\overline{\mathbb{C}} \setminus K_i)$ holds, we get by Theorem 2.1, $D(f_i) \leq d(K_i)$. Hence by using Proposition 3.1 we obtain the following

$$d(K) = \lim_{i \rightarrow \infty} d(L_i) \leq \lim_{i \rightarrow \infty} D(f_i) \leq \lim_{i \rightarrow \infty} d(K_i) = d(K),$$

which gives the desired limit (4.4). \square

Let K be a compact set in \mathbb{C}^n , and $J : A(K) \rightarrow C(K)$ the natural restriction homomorphism. $AC(K)$ is the Banach space obtained as the completion of the set $J(A(K))$ in the space $C(K)$ with respect to the uniform norm.

Lemma 4.6. Let K be an infinite polynomially convex compact set in \mathbb{C}^n . Then, for each bounded Borel measure $\mu \in \mathcal{M}(K)$, there exists an analytic functional $f^* \in A(K)^* \hookrightarrow A(\mathbb{C}^n)^*$ and a corresponding analytic germ $f' = T f^*$ such that

$$(4.5) \quad f^*(f) = \langle f, f' \rangle = \int_K f(\zeta) d\mu(\zeta),$$

for every $f \in A(\mathbb{C}^n)$.

Proof. The dense embedding $A(K) \hookrightarrow AC(K)$ implies, for the dual spaces, the following embedding: $AC(K)^* \hookrightarrow A(K)^*$. Since $AC(K)$ is a closed subspace of $C(K)$, every bounded Borel measure $\mu \in \mathcal{M}(K)$ defines a linear continuous functional $F^* \in AC(K)^*$ such that

$$F^*(f) = \int_K f(\zeta) d\mu(\zeta)$$

for every $f \in AC(K)$. Then, the restriction $f^* = F^*|_{A(K)}$ belongs to $A(K)^*$. By Lemma 2.2, since $A(K)^* \hookrightarrow A(\mathbb{C}^n)^*$, there is $f' \in A_0(\{\infty^n\})$ such that

$$f^*(f) = \langle f, f' \rangle = \left(\frac{1}{2\pi i} \right)^n \int_{\mathbb{T}_R^n} f(\zeta) \overline{f'(\zeta)} d\zeta, \quad f \in A(\mathbb{C}^n),$$

where \mathbb{T}_R^n is defined as in (2.5), and R is sufficiently large. \square

Now we show that, for any real compact set in \mathbb{C}^n , the equality in the estimate (2.7) is attained at some $f^* \in A(K)^*$.

Theorem 4.7. *Let $K \subseteq \mathbb{R}^n \subseteq \mathbb{C}^n$ be a compact set. Then $K \in (SSP)$.*

Proof. By Theorem 2.5 and Remark 2.6, there exists a measure $\mu \in \mathcal{M}(K)$ such that (K, μ) satisfies the Bernstein-Markov inequality. Let f^* be an analytic functional corresponding to μ by Lemma 4.6. Initially, we show that $Z_s(K, \mu) = m_s(n)! |H_{m_s(n)}(f^*)|$, where $Z_s(K, \mu)$ and $H_{m_s(n)}(f^*)$ are defined in Section 2. Indeed, considering the relation (2.8) gives :

$$(4.6) \quad m_s(n)! |H_{m_s(n)}(f^*)| = |f_{\zeta^{(m_s(n))}}^* (\dots (f_{\zeta^{(1)}}^* ([V(\zeta^{(1)}, \dots, \zeta^{(m_s(n))}]^2) \dots))|,$$

Since K is a real subset and so $[V(\zeta^{(1)}, \zeta^{(2)}, \dots, \zeta^{(m_s(n))})]^2$ is nonnegative, by iterating (4.5) $m_s(n)$ times, the righthand side of (4.6) becomes :

$$\int_K \dots \int_K |V(\zeta^{(1)}, \zeta^{(2)}, \dots, \zeta^{(m_s(n))})|^2 d\mu(\zeta^{(1)}) \dots d\mu(\zeta^{(m_s(n))}),$$

which is equal to $Z_s(K, \mu)$. Since $(m_s(n)!)^{\frac{1}{2l_s(n)}} \rightarrow 1$ as $s \rightarrow \infty$, we have, by Theorem 2.5,

$$d(K) = \lim_{s \rightarrow \infty} Z_s(K, \mu)^{\frac{1}{2l_s(n)}} = \lim_{s \rightarrow \infty} |H_{m_s(n)}(f^*)|^{\frac{1}{2l_s(n)}} = D(f').$$

\square

Problem. Characterize compact sets in \mathbb{C}^n such that either (SP) or (SSP) holds.

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